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Anisotropic Plane Symmetric Magnetized Model with Cosmological Constant

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Abstract In this paper, we have obtained the solutions of perfect fluid cosmological model in Cylindrically-symmetric space time (Marder in Proc. R. Soc. A 246:133, 1958) with varying cosmological constant in the presence of electromagnetic field. To get determinate model of the universe we assumed that the scalar of expansion in the model is proportional to the eigen-value of the shear tensor which lead to the condition $A = (BC)^n$. The magnetic field is due to an electric current produced along x-axis. Thus the magnetic field is in yz-plane and F_{23} is the only non-vanishing component of electromagnetic field tensor F_{ij}. Various physical and geometrical features of the model have been discussed.

Keywords Electromagnetic field · Perfect fluid · Cosmological constant Λ · Cosmology

1 Introduction

The basic role of cosmological constant is related to the observational evidence of high red-shift Type Ia supernovae (Riess et al. [2], Perlmutter et al. [3]) for a small decreasing value of cosmological constant ($\Lambda_{present} \leq 10^{-56}$ cm^{-2}) at the present epoch. In general relativistic quantum field theory, the cosmological term (Λ) interpreted as the energy density of the vacuum (Zeldovich [4, 5] and Fulling et al. [6]). The Λ term has also been interpreted in terms of the Higgs scalar field by Bergmann [7], Linde [8] studied that the cosmological term Λ is a function of temperature and is related to the process of broken symmetries. In modern cosmological theories the cosmological constant Λ remains a focal point of interest.

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A wide range of observations now suggest that the universe possesses a nonzero cosmological constant (Krauss and Turner [9]). The cosmological models without the cosmological constant are unable to explain satisfactorily problems like structure formation and the age of the universe (Singh et al. [10])

Recent interest in the cosmological constant term Λ have received considerable attention among researchers for various concept (Frieman and Waga [11], Carlberg et al. [12], Silviera and Waga [13]). Some of the recent discussion on the cosmological constant "problem" and on cosmology with a time-varying cosmological constant by Ratra and Peebles [14], Dolgov et al. [15], Dolgov [16] , Sahni and Starobinsky [17] point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a "constant", however, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying cosmological constant Λ can be found. For these solutions, conservation of energy requires decrease in the energy density of the vacuum component to be compensated by a corresponding increase in the energy density of matter or radiation.

Magnetic field plays a vital role in the description of the energy distribution in the universe as it contains highly ionized matter. Strong magnetic fields can be created due to adiabatic compression in cluster of galaxies. Large scale magnetic fields give rise to anisotropies in the universe. It is believed that the presence of electromagnetic field could alter the rate of creation of particles in the anisotropic models. A cosmological model which contains a global magnetic field is necessarily anisotropic since the magnetic field vector specifies a preferred spatial direction. Also, electromagnetic field directly affects the expansion rate of the universe. The occurrence of magnetic fields on galactic scale is well-established fact today, and their importance for a variety of astrophysical phenomena is generally acknowledged as pointed out by Zeldovich et al. [18]. Harrison [19] has suggested that magnetic field could have a cosmological origin. The presence of primordial magnetic field of cosmological origin in the early stages of the evolution of the universe has been discussed by eminent author's viz. Misner et al. [20], Asseo and Sol [21], Melvin [22], Kim et al. [23], Wolfe et al. [24] and Kulsrud et al. [25]. Anisotropic magnetic field models have significant contribution in the evolution of galaxies and stellar objects. Bali and Ali [26] had obtained a magnetized cylindrically symmetric universe with an electrically neutral perfect fluid as the source of matter. Cosmological models with an incident magnetic field for different space times have been investigated by several authors viz. Tupper [27], Roy and Prakash [28], Lorentz [29], Bali and Tyagi [30], Pradhan and Pandey [31], Pradhan and Singh [32], Katore and Rane [33], Wang [34].

Bali et al. [35] and Bali and Pareek [36] have studied Bianchi-I and III magnetized massive string perfect fluid cosmological models in General Relativity. In general relativity, the behavior of string in cylindrically symmetric inhomogeneous cosmological models in presence of electromagnetic field have been studied by Pradhan et al. [37]. Pawar et al. [38] have investigated plane symmetric string dust magnetized cosmological model with bulk viscous fluid in general relativity. Saha and Visinescu [39] have studied Bianchi-I string cosmological models in presence of magnetic flux and concluded that the presence of magnetic field results in a rapid growth of the volume scale factor. Bali and Pareek [40] have investigated Bianchi-V massive string cosmological models with free gravitational field of Petrov type degenerate in the presence of magnetic field with variable magnetic permeability. Tripathy et al. [41] have studied the effect of bulk viscosity and magnetic field upon an spatially homogeneous and anisotropic LRS Bianchi-I universe filled with string in the frame work of general relativity. Verma and Shri Ram [42] have investigated Bianchi-III anisotropic cosmological model with varying cosmological and gravitational constants in presence of bulk viscous fluid.

In this paper we have investigated the solutions of Einstein field equations in cylindrically- symmetric space time when the source of the gravitational field is governed by perfect fluid with varying cosmological constant in presence of electromagnetic field. The magnetic field is due to an electric current produced along x-axis. Thus the electromagnetic field is in yz-plane and F_{23} is the only non-vanishing component of electromagnetic field tensor F_{ij}. The behavior of the electromagnetic field tensor together with various physical and geometrical aspects of the model are discussed. At the end we summarize our conclusion.

2 The Metric and Field Equations

A cylindrically-symmetric metric is considered in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2 \tag{1}$$

where the metric potentials A, B and C are functions of t only. This ensures that the model is spatially homogeneous and anisotropic. The metric (1) can be transformed to the Bianchi type-I form by the coordinate transformation $t \to \int A(t)dt$. However, for convenience of mathematical calculations we retain the form (1) in this paper. The energy momentum tensor for perfect fluid with electromagnetic field is given by

$$T_i^j = (\rho + p)u_i u^j + p g_i^j + E_i^j, \tag{2}$$

where E_i^j is the electromagnetic field given by Lichnevowicz [43]

$$E_i^j = \bar{\mu}\left[|h|^2\left(u_i u^j + \frac{1}{2}g_i^j\right) - h_i h^j\right] \tag{3}$$

with $\bar{\mu}$ is the magnetic permeability and h_i the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}} \in_{ijkl} F^{kl}u^j, \qquad |h|^2 = h_l h^l \tag{4}$$

where ρ is the energy density, p the isotropic pressure, F_{ij} is the electromagnetic field tensor, $\in_{ijkl}$ is the Levi-Civita tensor density and u^i is the four velocity vector satisfying the condition

$$u^4 u_4 = -1. \tag{5}$$

We assume that coordinates to be comoving so that

$$u^1 = u^2 = u^3 = 0 \quad \text{and} \quad u^4 = A^{-1}. \tag{6}$$

The incident magnetic field is taken along x-axis so that $h_1 \neq 0$, $h_2 = 0 = h_3 = h_4$.

Due to assumption of infinite electrical conductivity (Roy Maartens [44]) of the fluid, we get $F_{14} = 0 = F_{24} = F_{34}$.

The only non-vanishing component of F_{ij} is F_{23}.

The Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \quad \text{and} \quad F^{ij}_{;j} = 0 \quad \text{i.e.} \quad \frac{\partial}{\partial x^j}(F^{ij}\sqrt{-g}) = 0 \tag{7}$$

together leads to the result

$$F_{23} = \text{constant} = H \ (\text{say}), \tag{8}$$

where the semicolon (;) represents a covariant differentiation. Hence

$$h_1 = \frac{AH}{\bar{\mu}BC}. \tag{9}$$

For the metric considered in (1), the components of electromagnetic field can be expressed as

$$E_1^1 = \frac{-H^2}{2\bar{\mu}B^2C^2} = -E_2^2 = -E_3^3 = E_4^4 \tag{10}$$

The Einstein's field equations

$$R_i^j - \frac{1}{2}Rg_i^j + \Lambda g_i^j = -8\pi T_i^j, \quad (c = 1,\ G = 1 \text{ in gravitational unit}) \tag{11}$$

for the metric (1) in presence of electromagnetic field with perfect fluid takes the forms:

$$\frac{1}{A^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right] + \Lambda = -8\pi\left[p - \frac{H^2}{2\bar{\mu}B^2C^2}\right], \tag{12}$$

$$\frac{1}{A^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} - \frac{\dot{A}^2}{A^2}\right] + \Lambda = -8\pi\left[p + \frac{H^2}{2\bar{\mu}B^2C^2}\right], \tag{13}$$

$$\frac{1}{A^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\dot{A}^2}{A^2}\right] + \Lambda = -8\pi\left[p + \frac{H^2}{2\bar{\mu}B^2C^2}\right] \quad \text{and} \tag{14}$$

$$\frac{1}{A^2}\left[\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right] + \Lambda = 8\pi\left[\rho + \frac{H^2}{2\bar{\mu}B^2C^2}\right]. \tag{15}$$

Here and in the following expressions dot (.) overhead letter denotes the ordinary differentiation with respect to t.

3 Solutions of field equations

Equations (12) to (15) are four equations connecting six unknowns A, B, C, p, ρ and Λ. For the complete determination of these field equations two more conditions are required. Here we assume the relation between metric potentials as

$$A = (BC)^n; \quad n \neq 1, \tag{16}$$

where n is a constant. Katore and Rane [33] obtained a magnetized cosmological models in bimetric theory of gravitation with the above condition.

From (12) and (13) we get

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = 0. \tag{17}$$

Using (17) in (12), (13) and (14) we get

$$\frac{1}{A^2}\left[\frac{\ddot{B}}{B}-\frac{\ddot{A}}{A}+\frac{\dot{A}^2}{A^2}-\frac{\dot{A}\dot{B}}{AB}-\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}\right]=8\pi\left[\frac{H^2}{\bar{\mu}B^2C^2}\right]. \tag{18}$$

From (16) and (18) we have

$$\frac{\ddot{B}}{B}+\frac{\dot{B}\dot{C}}{BC}=\frac{8\pi H^2}{(1-2n)\bar{\mu}}(BC)^{2n-2}. \tag{19}$$

Setting $BC=\alpha$ and $\frac{B}{C}=\beta$ so that $B^2=\alpha\beta$ and $C^2=\frac{\alpha}{\beta}$ then (17) and (19) goes to the form

$$\left(\frac{\alpha\ddot{\beta}}{\beta}\right)=0 \tag{20}$$

and

$$\ddot{\alpha}+\left(\frac{\alpha\dot{\beta}}{\beta}\right)=\frac{16\pi H^2}{(1-2n)\bar{\mu}}\alpha^{2n-1}. \tag{21}$$

From (20) and (21) one has

$$\alpha=[m(k_1t+k_2)]^{1/m}, \tag{22}$$

where k_2 is constant of integration and $m=1-n$.

Using (22) in (20), we get

$$\beta=k_3\exp\left\{\frac{-mb}{nk_1}[m(k_1t+k_2)]^{\frac{-n}{m}}\right\},$$

where b and k_3 are constants of integration.

Since $A=(BC)^n=\alpha^n=[m(k_1t+k_2)]^{\frac{n}{m}}$,

$$B^2=\alpha\beta=k_3[m(k_1t+k_2)]^{\frac{1}{m}}\cdot\exp\left\{\frac{-mb}{nk_1}[m(k_1t+k_2)]^{\frac{-n}{m}}\right\}\quad\text{and}$$

$$C^2=\frac{\alpha}{\beta}=\frac{1}{k_3}[m(k_1t+k_2)]^{\frac{1}{m}}\cdot\exp\left\{\frac{mb}{nk_1}[m(k_1t+k_2)]^{\frac{-n}{m}}\right\}.$$

Hence, the geometry of the space-time (1) takes the form

$$\begin{aligned} dS^2&=[m(k_1t+k_2)]^{\frac{2n}{m}}(dx^2-dt^2)+k_3[m(k_1t+k_2)]^{\frac{1}{m}}\exp\left\{\frac{-mb}{nk_1}[m(k_1t+k_2)]^{\frac{-n}{m}}\right\}dy^2\\ &\quad+\frac{1}{k_3}[m(k_1t+k_2)]^{\frac{1}{m}}\exp\left\{\frac{mb}{nk_1}[m(k_1t+k_2)]^{\frac{-n}{m}}\right\}dz^2. \end{aligned} \tag{23}$$

By suitable transformation (23) reduces to a simpler form

$$dS^2=T^{\frac{2n}{m}}(dX^2-dT^2)+T^{\frac{1}{m}}\cdot\exp\left(\frac{-mb}{nk_1}T^{\frac{-n}{m}}\right)dY^2+T^{\frac{1}{m}}\cdot\exp\left(\frac{mb}{nk_1}T^{\frac{-n}{m}}\right)dZ^2 \tag{24}$$

4 Some Physical and Geometrical Features of the Model

The energy density ρ, the effective pressure p for the model (24) are given by

$$8\pi\rho = \frac{1}{T^{\frac{2n}{m}}}\left[\frac{5}{4m^2T^2} - \frac{1}{mT^2} - \frac{b^2}{4k_1^2T^{\frac{2}{m}}} - \frac{4\pi H^2}{\bar{\mu}T^2}\right] + \Lambda \tag{25}$$

and

$$8\pi p = \frac{1}{T^{\frac{2n}{m}}}\left[\frac{1}{4m^2T^2} - \frac{b^2}{4k_1^2T^{\frac{2}{m}}} - \frac{4\pi H^2}{\bar{\mu}T^2}\right] - \Lambda. \tag{26}$$

To find the explicit value of cosmological constant $\Lambda(t)$, we assume that the fluid obeys an equation of state of the form

$$p = \gamma\rho, \quad \text{where } \gamma \ (0 \le \gamma \le 1) \text{ is constant.} \tag{27}$$

Using (27) in (25) and (26), we obtain

$$\Lambda = \frac{1}{(\gamma+1)T^{\frac{2n}{m}}}\left[\frac{1-\gamma(1+4n)}{4m^2T^2} + \frac{b^2(\gamma-1)}{4k_1^2T^{\frac{2}{m}}} + \frac{4\pi H^2(\gamma-1)}{\bar{\mu}T^2}\right] \tag{28}$$

The reality conditions (i) $\rho + p > 0$, (ii) $\rho + 3p > 0$ are given by Ellis [45] lead to

$$\frac{1+2n}{2m^2T^2} > \frac{b^2}{2k_1^2T^{\frac{2}{m}}} + \frac{8\pi H^2}{\bar{\mu}T^2} \quad \text{and} \tag{29}$$

$$\frac{2n}{m^2T^2} > \frac{b^2}{k_1^2T^{\frac{2}{m}}} + \frac{16\pi H^2}{\bar{\mu}T^2} + 2\Lambda T^{\frac{2n}{m}} \quad \text{respectively.} \tag{30}$$

The dominant energy conditions (Hawking and Ellis [46]) (i) $\rho - p \ge 0$, (ii) $\rho + p \ge 0$ lead to

$$\frac{n}{m^2T^2} \ge -2\Lambda T^{\frac{2n}{m}} \tag{31}$$

and

$$\frac{1+2n}{2m^2T^2} \ge \frac{b^2}{2k_1^2T^{\frac{2}{m}}} + \frac{8\pi H^2}{\bar{\mu}T^2} \quad \text{respectively.} \tag{32}$$

The conditions (30) and (31) impose the restriction on cosmological constant Λ. From (28), it is observed that the cosmological constant Λ is a decreasing function of time. Also from (25) and (26) we get, $\rho = \Lambda$ and $p = -\Lambda$ at large value of T, which implies $\rho + p = 0$. The matter density $\rho$and pressure p becomes infinite at $T \to 0$.

From (19) the non-vanishing component F_{23} of the electromagnetic field tensor is obtained as

$$F_{23} = \frac{1}{4m}\sqrt{\frac{\bar{\mu}n}{\pi}(1-2n)} = \text{constant.}$$

This shows that electromagnetic field tensor is constant with time for $n \neq 1$.

The expressions for the spatial volume V^3, Expansion scalar θ and the Shear scalar σ^2 for the model (24) are given by

$$\text{Spatial Volume:} \quad V^3 = T^{\frac{2n+1}{m}}. \tag{33}$$

$$\text{Expansion Scalar:} \quad \theta = U^a_{;a} = \frac{1}{mT}\left[n + \frac{1}{T^{\frac{n}{m}}}\right] \tag{34}$$

$$\text{Shear Scalar:} \quad \sigma^2 = \frac{1}{6m^2T^2}\left[n + \frac{1}{T^{\frac{n}{m}}}\right]^2. \tag{35}$$

From (34) and (35) we obtain $\frac{\sigma^2}{\theta^2} = \frac{1}{6} = \text{constant}$.

The model (24) represents an expanding, shearing and non-rotating universe. We observed that at large value of T the spatial volume increases, while the expansion scalar and shear scalar diverge at $T = 0$. Also since $\lim_{T\to\infty}(\frac{\sigma^2}{\theta^2}) = \frac{1}{6} \neq 0$, the model (24) does not approach isotropy for large T.

The rate of expansion H_i in the direction of x, y and z are given by

$$H_x = \frac{\dot{A}}{A} = \frac{n}{mT}, \tag{36}$$

$$H_y = \frac{\dot{B}}{B} = \frac{1}{2}\left[\frac{1}{mT} + \frac{b}{k_1 T^{\frac{1}{m}}}\right], \tag{37}$$

$$H_z = \frac{\dot{C}}{C} = \frac{1}{2}\left[\frac{1}{mT} - \frac{b}{k_1 T^{\frac{1}{m}}}\right]. \tag{38}$$

5 Discussion

In the present paper, we have obtained the solutions of perfect fluid cosmological model in cylindrically symmetric space time with varying cosmological constant term Λ in the presence of electromagnetic field. The source of magnetic field is due to an electric current produced along the x-axis. Thus the magnetic field is in yz-plane and F_{23} is the only non-vanishing component of electromagnetic field tensor F_{ij}. The electromagnetic field tensors do not vanish if $k_1 \neq 0$. In presence of electromagnetic field the model (24) of the universe represents an expanding, shearing and non-rotating universe. The cosmological constant Λ is decreasing function of time and approach a small value as time increases. Since $\lim_{T\to\infty}(\frac{\sigma}{\theta}) \neq 0$. Hence the anisotropy is maintained for large values of T.

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